EQUATIONS OF MOTION AND RADIATION REACTION

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SUMMARY

The Einsteinian field equations and the equations of motion are expanded into powers of the gravitational coupling constant. Starting from the equations for continuously distributed matter we arrive at the equations of motion for a system of point-like particles. We deal with the first and second approximations. Finally we get the equations of motion for point-like particles free from infinities and containing the radiation reaction terms.

The discussion, whether gravitational radiation exists or does not exist, is not yet finished. Various people have various opinions Einsteinian equations have wave-like solutions as well as the exact equations as well as approximate ones. Whether these waves have a real physical meaning or not, it is connected with the question, whether they transfer energy or not. To decide this question is very difficult, because the definition of energy in general relativity is not yet finally solved. A usable method to decide the question about the existence of gravitational radiations, it seems to us, is the investigation of the equations of motion of a system of interacting particles. From the kind of the motion one should conclude, whether the particles radiate energy to each other or not.

This program was suggested by Havas [1]. To deal with the full Einsteinian theory is mathematically too difficult. Therefore one needs approximat methods. The EIH-method is not suitable in this case, because radiation terms appear only in higher orders. Havas therefore used the fast motion approximation. He gets, besides of the expression for the self-energy, an equation of motion with radiation damping terms:

$$m_{A,ren}\ddot{a}^{\alpha} + \frac{11}{3}\frac{\gamma m_{\wedge}}{c^2}(\dddot{a}^{\alpha} + \dot{a}^{\alpha}\dot{a}^{2}) +$$

+ force originated by the other mass points = 0 (the dot means $\frac{d}{d\tau}$, τ is the Minkowskian proper time, γ is the Newtonian gravitational constant).

With this equations of motion Havas and Smith calculated the motion of two masses around each other. They found a spiral outward. It would mean, that a moving mass in gaining energy by the gravitational radiation. This result is not yet fully convincing. Up to the first order the equations of motion contain the velocities in zeroth-order (it means $\dot{v}^\alpha = 0$).

The radiation damping terms are proportional to ε ($\varepsilon = 4\gamma c^{-2}$) the influence of the other masses on the velocity, too. Therefore one wants to use the second order equation of motion, which contain the velocities in first order. With this equation one could calculate the motion of two mass points (two-body-problem) and then could conclude reliably on the existence or non-existence of the gravitational radiation.

For this purpose we need the equations of motion for a system of point-like particles up to second order in ε without infinities. To remove these infinities after the application of δ-functions (s. e. g. the paper of Bertotti and Plebanski [2]) has not been reached up with the methods suitable for the linear approximation; either the definition of energy is needed or the mathematical difficulties are not overcome.

We will deduce the equations of motion for a system of point-like particles by the transition from extended fluid-drops to point-like particles. We start from the field equations for a perfect compressible fluid and use de Donder's coordinate condition. Then we expand the $\mathfrak{g}^{\alpha\beta}$($\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$) instead of the metric quantities $g_{\alpha\beta}$.

We do not accept a definite equation of state, but the pressure shall be a unique function of the mass density. For we deal with a finite-sized fluid in its gravitational field, there exists a condition of stability for this "drop". From this condition we conclude, that the expansion of the pressure starts with a term of first order in ε.

The field equations in the linear approximation are

$$\Box \underset{1}{\mathfrak{g}}^{\alpha\beta} = -4\pi\varepsilon\mu v^\alpha v^\beta$$

($\Box$ is the d'Alembertian, μ is the mass density). The solution is:

$$\underset{1}{\mathfrak{g}}^{\alpha\beta}(x) = \varepsilon \int\limits_{\Omega} \mu(x')v^\alpha(x')v^\beta(x')D(x-x')d_4x'.$$

Ω is the fourdimensional volume of the fluid, D is the retarded Green's function with $\Box D(x) = -4\pi\delta_4(x)$.

The equation of motion up to the first order in ε is

$$\mu\ddot{a}^\alpha + \frac{1}{2}\mu(\eta^{\alpha\varrho} - \dot{a}^\alpha\dot{a}^\varrho)\left\{\varepsilon\int\limits_{\Omega}\left[\left((\dot{a}\dot{b})^2 - \frac{1}{2}\right)D_{1\varrho}(a-b) - 2\dot{b}_\varrho(\dot{a}\dot{b})\frac{d}{d\tau_a}D(a-b)\right]\cdot\right.$$
$$\left.\cdot\,\mu(b)d_4b - \frac{1}{c^2}p_{1\varrho}\right\} = 0 \tag{1}$$

$$\text{with} \quad \eta_{\alpha\beta}\dot{a}^{\alpha}\,\dot{a}^{\beta}=\dot{a}^{2}=1$$

We imagine the fluid consisting of drops separated from each other. We investigate the equation of motion for one of these drops and single out the terms from which infinities may arise after the transition to point-like particles.

We make the following assumptions: A drop can be described by one unique velocity; we do not consider internal motions (rotations, turbulence, deformations). This velocity does not depend on the spatial coordinates. We imagine the drops to be small spheres, the mass distribution shall be spherically symmetric.

For a system of drops separated from each other the integral in equ. (1) becomes a sum of integrals over the regions of the single drops. First we integrate this equation over the region of the A-th drop, to introduce the mass m_A. Only the one term, the ranges of integrations is which coincide, will yield infinite contributions. From the other terms of the sum one gets the same contribution as Bertotti and Plebanski found with help of δ-functions.

We transform the interesting part of equ. (1) into a more suitable form and make then an expansion analogous to those in Dirac's paper [3]. Finally we get an equation of stability, a mass renormalization terms

$$\delta m_A = -\frac{7}{8}\,\varepsilon \int_A \mu(a)d_3a \int_A \mu(b)d_3b\cdot\lambda^{-1}, \quad \lambda = |a-b|,$$

a Havas' radiation reaction terms.

The linear approximation was only a test for our method.

The equation of motion up to second order in ε is a rather long equation. It contains already the non-linear features of the Einsteinian theory. There appear products of two and three D-functions. We can apply our method in this case and get a further mass renormalisation of the coupling constant $(\varepsilon\, m_A^2)$ before the radiation reaction terms of the first order. The equation of stability is much more involved than that in first order. We need not only a scalar pressure but a stress tensor, therefore we must leave the perfect fluid model in some respects. But the choice of this special model does not influence the results we are interested in which up to the second order in ε.

The remaining and after renormalised finite part of the equations of motion coincides with the equation of Bertotti and Plebanski for point-like particles and derived with help of δ-functions. In addition there appears a terms like a radiation reaction terms: $-\dfrac{253}{192}\,(\ddot{a}^{\alpha}+3\dot{a}^{\alpha}\dot{a}\overset{\cdot\cdot\cdot\cdot\cdot}{a})$. This terms is, however, not due tot he radiation field, because it is invariant with respect to a time reversal.

In this way we have got the equation of motion up to the second order fast motion approximation free from infinities, and we have seen that no new radiation reaction force is appearing in the second order equation.

To decide the question about the existence of gravitational radiation, from our point of view it is necessary to calculate a special motion, say an elliptic orbit, and to find the derivations from this orbit produced by the gravitational radiation. But it is still a future program.

LITERATURE

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